

1952

Equivalent functions of strategies in the theory of games

Harlan D. Mills
Iowa State College

Follow this and additional works at: <https://lib.dr.iastate.edu/rtd>



Part of the [Mathematics Commons](#)

Recommended Citation

Mills, Harlan D., "Equivalent functions of strategies in the theory of games " (1952). *Retrospective Theses and Dissertations*. 12858.
<https://lib.dr.iastate.edu/rtd/12858>

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.

**EQUIVALENT FUNCTIONS OF STRATEGIES
IN THE THEORY OF GAMES**

by

Harlan D. Mills

**A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY**

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

1952

UMI Number: DP11920

INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

UMI[®]

UMI Microform DP11920

Copyright 2005 by ProQuest Information and Learning Company.

All rights reserved. This microform edition is protected against unauthorized copying under Title 17, United States Code.

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

TABLE OF CONTENTS

INTRODUCTION	1
The Zero-sum Two-person Game	1
Properties of Games	5
Summary	9
A BILINEAR EXPANSION	11
Derivation	11
Comparison with Analytic Expansions	18
PUNISHMENT, LIMBO, AND REWARD	20
Simple Games	20
Artificial Games	22
Homogeneity and Convexity in Punishment, Limbo and Reward	26
Solutions for Regular Games	33
AN EQUIVALENCE RELATION FOR ALL GAMES	37
Equivalence of A and TA	37
Invariants of $T(A)$	41
Some Special Classes	44
METHODS AND CONJECTURES	46
Methods	46
Conjectures	48
LITERATURE CITED	51
ACKNOWLEDGMENT	52

INTRODUCTION

The Zero-sum Two-person Game

Consider a contest among a set of players, where the skill of the players and, possibly, chance events effect the outcome. Call the set of all rules of the contest the game, and any particular course the contest may take, from beginning to end, a play. Suppose the game prescribes, for any play, payments to the players. Profits are denoted by positive payments, and losses shown by negative payments.

It is assumed that at most a finite number of situations can arise in the contest, and, at each situation, at most a finite number of alternatives is available to the player or chance device making the choice in this situation. Then, each of the players can specify, previous to a play of the contest, exactly which choice will be made for every possible situation in which he might have an option. All information allowable under the game can be included as a basis for a choice, so this pre-specification entails no disadvantage to the player. A complete --exhaustive and unambiguous-- set of such choices is called a strategy, and, under the assumptions made above, there is at most a finite number of strategies for each player.

If chance events may effect the outcome of the contest, consider all possible situations where a chance device may

make a choice, and select one of these choices arbitrarily. Call a complete set of such choices an umpire's choice, and, from the assumptions above, there is at most a finite number of umpire's choices. Let the probabilities for any chance device's making its choices be known. Then, the probability of occurrence for any umpire's choice can be found.

If each of the players selects a strategy, and an umpire's choice is chosen, a play of the contest can be constructed, and payments assigned to the players. Since an umpire's choice has a definite probability of occurrence, all possible umpire's choices can be considered, for a fixed set of players' strategies, and the mathematical expectation of the payments can be calculated. Thus, for each set of strategies selected by the players, a set of payments (in the sense of expectation) can be assigned the players.

When the contest is between two players, its rules are called a two-person game. When the sum of the payments is zero for every possible play under the game, it is called a zero-sum game. A zero-sum two-person game describes a contest between two players where one player wins exactly the same payment that the other player loses. Each player selects a strategy, and the game then specifies a payment which one player must make to the other.

Let players P_x and P_y enter a contest under a two-person zero-sum game. Suppose P_x has m possible strategies,

labeled $1, 2, \dots, m$, and P_y has n possible strategies, called $1, 2, \dots, n$. Let the payment to P_x be p_x , and the payment to P_y be p_y . Call the set of all strategies for each player M or N respectively, with a typical element i or j . That is, reading " \in " as "is (be) an element of", one can write

$$i \in M = (1, 2, \dots, m), j \in N = (1, 2, \dots, n).$$

Then, the description above becomes,

$$p_x = p_x(i, j), p_y = p_y(i, j), p_x(i, j) = - p_y(i, j).$$

For convenience, define,

$$a_{ij} = p_x(i, j) = - p_y(i, j), i \in M, j \in N, \\ A = (a_{ij}).$$

The domains of i and j are independent of each other, and A is an m by n matrix, called a payoff matrix. Any row of A identifies a strategy selection by P_x , while any column of A represents a selection of a strategy by P_y . If both players want to realize a most favorable outcome, P_x will wish to maximize, and P_y will want to minimize, the entry of A which is ultimately selected by the intersection of the row and column chosen by each of the players.

In a certain rational sense (1, pp. 98), a theory can be developed for some, but not all, games in the formulation given above. However, the adoption of a decisive new concept in strategy selection will permit a rational theory which includes all zero-sum two-person games.

The consideration of certain simple games, such as

matching pennies (1, pp. 143), shows that a player may be outwitted and punished if he is required to select, on the basis of his own reasoning, a particular strategy for a play of a contest. In view of this it is demanded only that a player state the probability with which he will play a particular strategy. That is, instead of selecting a strategy, let the player select a probability for playing each of all possible strategies. Call the probability of P_x playing the i th strategy x_i , and the probability of P_y playing the j th strategy y_j . The collection of x_i is a vector in E_m , Euclidean m space, and the collection of y_j is a vector in E_n , say

$$x = (x_i), y = (y_j).$$

If the players select the sets of probabilities, x and y , for playing their strategies, the expected value for the payoff will be

$$xAy = x_i a_{ij} y_j, \quad (1)$$

where the convention of summing repeated indices through their domains of definition is adopted.

The sets x and y must be sets of mutually exclusive and exhaustive probabilities. Use the definitions,

$$e_i = 1 \text{ for all } i, e = (e_i), E = (e_{ij}) = (e_i e_j).$$

Then, the conditions on x and y become

$$xe = x_i e_i = 1, ey = e_j y_j = 1, \quad (2)$$

$$x_i \geq 0 \text{ for all } i, y_j \geq 0 \text{ for all } j. \quad (2a)$$

The vectors x and y are called mixed strategies. Let X be the set of all x satisfying (2) and (2a), and Y be the set of all y satisfying (2) and (2a). The unit vectors of E_m and E_n are elements of X and Y respectively. These unit vectors are called pure strategies, and represent an actual row or column selection of A . Hence, mixed strategies include all the previous notions on strategy selection. It turns out that the concept of mixed strategies will permit the formulation of a rational theory for the playing of any zero-sum two-person game.

Properties of Games

It is assumed that the reader is acquainted with the notions of geometric convexity (1, pp. 128) and the functional operators Max_x and Min_x (1, pp. 88).

A statement of the utmost importance in the theory of games is due originally to John von Neumann. It is stated here without proof (1, pp. 153).

Theorem. (Main Theorem for zero-sum two-person games.)

Let A be an arbitrary real matrix. Then there exist vectors, $x^0 \in X$, $y^0 \in Y$, and a real number, Δ , such that for every $x \in X$, $y \in Y$,

$$x^0 A y \geq \Delta \geq x A y^0. \quad (3)$$

In game terms, this theorem asserts that P_x has a mixed strategy, x° , which will guarantee a payoff of not less than Δ , while P_y has a mixed strategy, y° , which will guarantee a payoff of not more than Δ . Thus, Δ is a point of impasse, beyond which neither player need be allowed by an opponent. This point of impasse, Δ , is called the value of the game A . A strategy, x° or y° , which guarantees that an opponent will not be allowed a more favorable payoff than Δ is called an optimal strategy. A pair of such strategies, x° and y° is called a solution. Let the set of all x° be X° , the set of all y° be Y° . To emphasize the dependence of Δ , X° , Y° on A , they may be written, $\Delta(A)$, $X^\circ(A)$, $Y^\circ(A)$.

Using these definitions, the Main Theorem states that every zero-sum two-person game has a value, and each player has an optimal strategy which guarantees him that value. In a certain sense, rational playing implies the use of the optimal strategies, and the value of the game represents a measure of the utility of the game itself to the players. A game which has zero value is called a fair game.

The main problem in the theory of the zero-sum two-person games at the present time is to find methods for obtaining the value and sets of optimal strategies for a given game.

Some well known properties of games are given here, with demonstrations when space permits, and references otherwise.

Property 1. The Main Theorem can be given in the following equivalent forms.

A. Let A be arbitrary. Then there exist vectors, $x^\circ \in X$, $y^\circ \in Y$, and a number, Δ , such that,

$$x^\circ_i a_{ij} \geq \Delta \text{ for all } j, a_{ij} y^\circ_j \leq \Delta \text{ for all } i. \quad (3A)$$

B. Let A be arbitrary, and the domain of x be X , the domain of y be Y . Then, xAy has a saddle point. That is, there exist vectors $x^\circ \in X$, $y^\circ \in Y$ such that,

$$x^\circ Ay \geq x^\circ Ay^\circ \geq xAy^\circ \text{ for every } x \in X, y \in Y. \quad (3B)$$

C. Let A be arbitrary, and the domain of x be X , the domain of y be Y . Then,

$$\text{Max}_x \text{Min}_y xAy = \text{Min}_y \text{Max}_x xAy. \quad (3C)$$

Property 2. X° and Y° are convex hyper-polyhedra.

Property 3. Let A and B be two games, γ be a real number.

A. Let $B = A + \gamma E$. Then, $\Delta(B) = \Delta(A + \gamma E) = \Delta(A) + \gamma$, $X^\circ(B) = X^\circ(A)$, $Y^\circ(B) = Y^\circ(A)$.

B. Let $\gamma > 0$, $B = \gamma A$. Then, $\Delta(B) = \Delta(\gamma A) = \gamma \Delta(A)$, $X^\circ(B) = X^\circ(A)$, $Y^\circ(B) = Y^\circ(A)$.

Demonstration of Property 1A. Let $x^\circ \in X^\circ$ and Δ be chosen by means of the Main Theorem. If the relation,

$$x^\circ_i a_{ij} \geq \Delta \text{ for all } j,$$

fails, suppose it fails for $j = k$. Let y' be defined as

$y'_k = 1, y'_j = 0$ for $j \neq k$. Then $y' \in Y$, and

$$x^\circ Ay' = x^\circ_1 a_{1j} y'_j = x^\circ_1 a_{1k} \not\geq \Delta.$$

But the statement of the Main Theorem is precisely the relation denied above, hence the property is shown as it regards to x . A symmetric argument will apply for y .

Demonstration of Property 1B. In the Main Theorem, let $x = x^\circ \in X^\circ, y = y^\circ \in Y^\circ$. Then, (3) becomes,

$$x^\circ Ay^\circ \geq \Delta \geq x^\circ Ay^\circ, \text{ or } \Delta = x^\circ Ay^\circ.$$

Substituting this relation back into (3), one obtains (3B), which defines a saddle point (1, p. 95).

Reference for Property 1C. (1, pp. 95).

Demonstration of Property 2. Only convexity is shown.

Let $x^\circ \in X^\circ, x^{\circ'} \in X^\circ$. Then it must be shown that when

$0 \leq \alpha \leq 1$, then $x^{\circ''} = \alpha x^\circ + (1 - \alpha)x^{\circ'} \in X^\circ$. Let $y \in Y$. Then

$$\begin{aligned} x^{\circ''}Ay &= (\alpha x^\circ + (1 - \alpha)x^{\circ'})Ay, \\ &= \alpha x^\circ Ay + (1 - \alpha)x^{\circ'}Ay, \\ &\geq \alpha \Delta + (1 - \alpha)\Delta = \Delta. \end{aligned}$$

Hence, $x^{\circ''} \in X^\circ$. A symmetric argument holds for Y° .

For the assertion that X° and Y° are hyper-polyhedra, see (2, pp. 3) and (2, p. 34). The statement appears to be plausible through an examination of the form of Property 1A. It is easy to see there, that X° , for instance, is the intersection of the $m + n$ half-spaces,

$$\begin{aligned} x_1 a_{1j} &\geq \Delta, \text{ for all } j, \\ x_i &\geq 0, \text{ for all } i, \end{aligned}$$

and the hyperplane, $ex = 1$.

Demonstration of Property 3A. Let $x^\circ \in X^\circ(A)$, $y^\circ \in Y^\circ(A)$. Then, for every $x \in X$, $y \in Y$,

$$x^\circ Ay \geq \Delta(A) \geq xAy^\circ.$$

Add γ to each member, and use the fact that for every $x \in X$, $y \in Y$, $xEy = (xe)(ey) = 1$. Then,

$$x^\circ Ay + \gamma x^\circ Ey \geq \Delta(A) + \gamma \geq xAy^\circ + \gamma xEy^\circ,$$

$$x^\circ(A + \gamma E)y \geq \Delta(A) + \gamma \geq x(A + \gamma E)y^\circ.$$

The last form demonstrates that $x^\circ \in X^\circ(B)$, $y^\circ \in Y^\circ(B)$, $\Delta(B) = \Delta(A) + \gamma$. If $x^\circ \in X^\circ(B)$, $y^\circ \in Y^\circ(B)$, a similar demonstration will show that $x^\circ \in X^\circ(A)$, $y^\circ \in Y^\circ(A)$.

Hence, the respective sets are identical.

Demonstration of Property 3B. Since $\gamma > 0$, the three members in the statement of the Main Theorem can be multiplied by γ without altering the inequalities. Then, the arguments used in Property 3A can be duplicated.

Summary

In the second chapter, a bilinear expansion is developed with a two-fold purpose. First, the expansion is an aid in the computation of optimal strategies and values for certain games. In addition, the expansion provides a basis for the classification and comparison of various games. While the derivation is algebraic in

character, the set of all expansion points is shown to be exactly the set of all extremal points of the constrained bilinear form obtainable by the method of LaGrange multipliers. However, additional freedom is obtained over analytic expansions by requiring that the expansion be identical to the bilinear function only in the intersection of the hyper-planes $e_x = 1$, and $e_y = 1$.

Some of the main results of the paper are obtained in the third chapter, where concepts of punishment, limbo, and reward are used to develop criteria for playing a game. All games are divided into two classes by these concepts, and a complete description of the solutions and value of any game can be given within the framework of these concepts.

An operator, which arises naturally in the bilinear expansion, is treated in the fourth chapter. This operator divides all games of a given size into equivalence classes, and the properties of these classes exhibit characteristics which seem pertinent to game theory.

In the fifth chapter, areas are pointed out which might be tractable with an extension of the analysis. Also, a game occurring elsewhere in the literature is analyzed, and the concepts of punishment, limbo, and reward are displayed in this example.

A BILINEAR EXPANSION

In this chapter, an expression, valid for a certain class of matrices, will be found which has functional values identical to xAy when x and y are restricted by (2). The conditions for existence for such an expression are found, and a comparison is made with analytic expansions.

Set notation will be used. In context, \emptyset is the null, or empty, set. The intersection of sets X and Y , written XY , implies that if $z \in XY$, then $z \in X$ and $z \in Y$.

Derivation

Consider the bilinear form,

$$xAy = x_i a_{ij} y_j, \quad i \in M, j \in N, \quad (1)$$

where x and y satisfy,

$$xe = 1, ey = 1. \quad (2)$$

Call the set of all x which satisfy (2) X' , and the set of all y which satisfy (2) Y' . Thus, X is a subset of X' , and Y is a subset of Y' .

By means of (2), define a vector, $x^p = (x^p_i)$, formally by replacing the component x_p in x by $(1 - x_\alpha e_\alpha)$, $\alpha \in M_p$, where M_p is the set M with p deleted. That is,

$$x^p_\alpha = x_\alpha, \quad x^p_p = 1 - x_\alpha e_\alpha, \quad \alpha \in M_p.$$

In a similar manner, using N_q as the set N with q deleted, define a vector, $y^q = (y^q_j)$, by,

$$y_{\beta}^q = y_{\beta}, y_q^q = 1 - e_{\beta} y_{\beta}, \beta \in N_q.$$

Also, let a set of numbers w_{pq} , $W = (w_{pq})$, $p \in M$, $q \in N$, satisfy,

$$e_{pq} w_{pq} = 1. \quad (4)$$

Lemma 1. Let $x \in X'$, $y \in Y'$. Then,

$$x Ay = w_{pq} x^p A y^q.$$

Proof. By the construction given, when $x \in X'$, $y \in Y'$,

$$x = x^p, y = y^q, x^p A y^q = x Ay.$$

Then,

$$w_{pq} x^p A y^q = e_{pq} w_{pq} x Ay = x Ay.$$

Lemma 2. A necessary and sufficient condition that

$$x^p A y^q = \rho + (x - u)H(y - v) \quad (5)$$

is that the following relations hold for $i \in M$, $j \in N$,

$$H = (h_{ij}), h_{ij} = a_{ij} - a_{iq} - a_{pj} + a_{pq}, \quad (6)$$

$$u = (u_i), u_i h_{ij} = a_{pq} - a_{pj}, \quad (7)$$

$$v = (v_j), h_{ij} v_j = a_{pq} - a_{iq}, \quad (8)$$

$$\rho + u_i h_{ij} v_j = a_{pq}. \quad (9)$$

Proof. First, write $x^p A y^q$ out as

$$\begin{aligned} x^p A y^q &= x_{\alpha} a_{\alpha\beta} y_{\beta} + (1 - x_{\alpha} e_{\alpha}) a_{p\beta} y_{\beta} + x_{\alpha} a_{\alpha q} (1 - e_{\beta} y_{\beta}) \\ &\quad + (1 - x_{\alpha} e_{\alpha}) a_{pq} (1 - e_{\beta} y_{\beta}), \alpha \in M_p, \beta \in N_q. \end{aligned}$$

Then, rearrange this expression, with the result,

$$\begin{aligned} x^p A y^q &= a_{pq} - x_{\alpha} (a_{pq} - a_{\alpha q}) - (a_{pq} - a_{p\beta}) y_{\beta} \\ &\quad + x_{\alpha} (a_{\alpha\beta} - a_{\alpha q} - a_{p\beta} + a_{pq}) y_{\beta}, \alpha \in M_p, \beta \in N_q. \end{aligned}$$

It is easy to verify the fact that the domains of α and β can be formally extended to include p and q respectively, for having done so, the coefficients of x_p and y_q become identically zero. Now, consider the form,

$$\begin{aligned} K &= \rho + (x_1 - u_1)h_{1j}(y_j - v_j), \\ &= \rho + u_1h_{1j}v_j - x_1h_{1j}v_j - u_1h_{1j}y_j + x_1h_{1j}v_j. \end{aligned}$$

K and $x^p A y^q$ are polynomials in the same variable (vacuously in x_p and y_q) and will be identical if and only if their coefficients are equated. Relations (6), (7), (8), and (9) are the statements of those equalities.

Relations (7) and (8) do not determine u_p or v_q . Their definitions are taken to be,

$$u_e = 1, \quad v_e = 1. \quad (10)$$

Let the set of all u satisfying (7) and (10) be U , and the set of all v satisfying (8) and (10) be V .

Lemma 3. Let $U \neq 0$, $V \neq 0$. Then, U , V , and ρ are all independent of the indices p and q . In addition, $u \in U$ if and only if u satisfies (10) and

$$u_i(a_{1j} - a_{1k}) = 0, \quad i \in M, \quad j \in N, \quad \text{any fixed } k \in N, \quad (7')$$

$v \in V$ if and only if v satisfies (10) and

$$(a_{1j} - a_{kj})v_j = 0, \quad i \in M, \quad j \in N, \quad \text{any fixed } k \in M, \quad (8')$$

and finally,

$$\rho = uAv, \quad u \in U, \quad v \in V. \quad (9')$$

Proof. If (6) and (10) are used in (7),

$$u_i(a_{ij} - a_{iq} - a_{pj} + a_{pq}) = a_{pq} - a_{pj},$$

$$u_i(a_{ij} - a_{iq}) + u_i e_i(a_{pq} - a_{pj}) = a_{pq} - a_{pj},$$

$$u_i(a_{ij} - a_{iq}) + a_{pq} - a_{pj} = a_{pq} - a_{pj}.$$

That is,

$$u_i(a_{ij} - a_{iq}) = 0, \quad i \in M, \quad j \in N.$$

Let $j = k$, any fixed element of N and subtract from above,

$$u_i(a_{ij} - a_{iq}) - u_i(a_{ik} - a_{iq}) = u_i(a_{ij} - a_{ik}) = 0.$$

All operations used in the reduction are reversible, hence (7') can be used with (6) and (10) to obtain (7), and (7') is thus equivalent to (7). The form of (7') shows that U is independent of p and q . A symmetric argument and reduction will demonstrate the assertions for V .

When $U \neq 0$, $V \neq 0$, the relation given by (5) exists, for h_{ij} and ρ are polynomial functions of a_{ij} , u_i , v_j and always exist formally. Let $x = u \in U$, $y = v \in V$. Then, $x \in X'$, $y \in Y'$, and the conditions of Lemma 1 are satisfied. Using this substitution, then, one obtains,

$$uAv = u^p A v^q = \rho + (u - u)H(v - v) = \rho.$$

This form shows that ρ is independent of p and q .

By Lemma 3, the entire dependence on p and q of the relation given in Lemma 2 is contained in H . With this in mind, let the notation for H be altered to $H_{pq} = (h_{pqij})$,

$$h_{pqij} = a_{ij} - a_{iq} - a_{pj} + a_{pq}. \quad (6')$$

An inspection of (6') shows that,

$$h_{pqij} = h_{ijpq} = -h_{pj iq} = -h_{iqpj},$$

and, as was noted in Lemma 2,

$$h_{pqpj} = h_{pqiq} = h_{pqpq} = 0.$$

Let a linear combination of H_{pq} over the indices p and q with coefficients from W restricted by (4) be written as,

$$D = w_{pq}H_{pq}, \quad d_{ij} = w_{pq}h_{pqij}. \quad (11)$$

Let f be an arbitrary vector in E_m , and g be an arbitrary vector in E_n . Then, two additional types of matrices are defined as,

$$F = (f_i e_j), \quad G = (e_i g_j). \quad (12)$$

F is a matrix of constant rows, while G has constant columns.

Lemma 4. Let $x \in X'$, $y \in Y'$, $u \in U$, $v \in V$, $C \in (E, F, G)$. Then

$$(x - u)C(y - v) = 0.$$

Proof. When $C = E$,

$$\begin{aligned} (x - u)E(y - v) &= (ex - eu)(ey - ev), \\ &= (1 - 1)(1 - 1) \\ &= (0)(0) = 0. \end{aligned}$$

When $C = F$, the right zero factor exhibited above will again be zero, and when $C = G$, the left zero factor will again be zero.

Let D , E , F , and G be defined as above, and γ be an arbitrary number. Then,

$$B = D + \gamma E + F + G \quad (13)$$

is a mapping of A which is a function of W , γ , f , and g .

Written out, an entry of B in (13) is,

$$b_{ij} = w_{pq}(a_{ij} - a_{iq} - a_{pj} + a_{pq}) + \gamma + f_i + g_j,$$

and is formalized as,

$$B = T(w_{ij}, \gamma, f_i, g_j)A = T(W, \gamma, f, g)A = TA.$$

Call the set of all matrices satisfying (13) $\mathcal{T}(A)$.

The reader may note that $\mathcal{T}(A)$ may be written as the set of all matrices of the form $A + F' + G'$, where F' and G' are arbitrary of the type defined in (12). However, the arguments of T will be seen ultimately to display certain characteristics of the matrix TA which seem pertinent to the investigation. For this reason, the first formulation is chosen, though it is, admittedly, more complex.

Theorem 1. Let $U \neq 0$, $V \neq 0$. If $u \in U$, $v \in V$, $B \in \mathcal{T}(A)$, and $x \in X'$, $y \in Y'$, then,

$$x Ay = u Av + (x - u)B(y - v). \quad (14)$$

Proof. Use Lemma 1, to obtain,

$$x Ay = w_{pq} x^p A y^q,$$

which can be written, by means of Lemma 2 and Lemma 3,

$$\begin{aligned} x Ay &= w_{pq}(\rho + (x - u)E_{pq}(y - v)), \\ &= u Av + (x - u)D(y - v). \end{aligned}$$

By Lemma 4, $\gamma E + F + G$ can be added vacuously to D . Hence,

$$x Ay = u Av + (x - u)B(y - v).$$

Lemma 5. $U \neq 0$ if and only if there exists no t satisfying,

$$At = e, et = 0. \quad (7e)$$

$V \neq 0$ if and only if there exists no s satisfying,

$$sA = e, se = 0. \quad (8e)$$

Proof. $U \neq 0$ if and only if the coefficient and augmented matrices in relations (7') and (10) have the same rank. No linear combination of the right side of (7') can give the right side of (10). Hence, $U \neq 0$ if and only if there exists no set of numbers, t_j , $j \in N$, such that,

$$(a_{ij} - a_{ik})t_j = 1 \text{ for all } i.$$

If such a set of numbers does exist, multiply these relations by t_k , and sum over k , with the result,

$$a_{ij}t_j e_k t_k - a_{ik} t_k e_j t_j = e_k t_k.$$

The left side of this is easily seen to be zero, hence,

$$et = 0.$$

Use this fact in the first expression, and obtain,

$$(a_{ij} - a_{ik})t_j = a_{ij}t_j - a_{ik}e_j t_j = 1 \text{ for all } i.$$

The second term of the center member is zero, and,

$$At = e.$$

Thus, if $U = 0$, a t exists, satisfying (7e). On the other hand, let t exist satisfying (7e), then,

$$(a_{ij} - a_{ik})t_j = a_{ij}t_j = 1,$$

and $U = 0$. Hence $U = 0$ if and only if there exists a t satisfying (7e), and the first statement of the Lemma follows immediately. A symmetric argument holds for V .

The statement of Lemma 5 gives conditions for the existence of the expression given in Theorem 1. When A is square, the conditions reduce to the fact that the sum of the cofactors of A must not be zero.

Comparison with Analytic Expansions

The form of the expression in Theorem 1 suggests a comparison with analytic expansions of xAy about extremal points in X' and Y' .

Lemma 6. $A \in T(A)$. In particular, let W satisfy (4). Then,

$$A = T(w_{ij}, -w_{pq}a_{pq}, e_p w_{pq}a_{iq}, w_{pq}e_q a_{pj})A.$$

Proof. A typical entry of an element of $T(A)$ is

$$\begin{aligned} b_{ij} &= w_{pq}(a_{ij} - a_{iq} - a_{pj} + a_{pq}) + \gamma + f_i + g_j, \\ &= a_{ij} + (-e_p w_{pq}a_{iq} + f_i) + (-w_{pq}e_q a_{pj} + g_j) \\ &\quad + (w_{pq}a_{pq} + \gamma). \end{aligned}$$

Now γ , f , and g are completely arbitrary. Thus, for any fixed W , the last three terms of the latter expression will vanish if a suitable set of definitions is taken for γ , f , and g . Such a set is displayed in the arguments of T in the statement of the theorem.

If the method of LaGrange multipliers is used to find extremal points in X' and Y' of xAy , with multipliers α and

β , the extremal points, u' and v' , satisfy these relations,

$$u'A = \alpha e, u'e = 1,$$

$$Av' = \beta e, ev' = 1.$$

Let the set of all such u' be U' , and the set of all such v' be V' . If $U' \neq 0$, $V' \neq 0$, the analytic expansion is,

$$x Ay = u' Av' + (x - u') A (y - v'), u' \in U', v' \in V'. \quad (14a)$$

Lemma 7. $U'(A) = U(A)$, $V'(A) = V(A)$.

Proof. Let $z \in U'(A)$. Then,

$$z_1(a_{ij} - a_{ik}) = \alpha - \alpha = 0, z e = 1,$$

hence, $z \in U(A)$. Now let $z \in U(A)$. Then (k is fixed),

$$z_1 a_{ij} = z_1 a_{ij} - z_1(a_{ij} - a_{ik}) = z_1 a_{ik} = \alpha, z e = 1,$$

hence, $z \in U'(A)$. A symmetric argument holds for V , V' .

It is interesting to note that the value of the multiplier, α , is $u' Av'$, the functional value of the extremal point.

Also by symmetry, $\beta = \alpha = u' Av' = u Av = \rho$.

Corollary 1. If an analytic expansion of the type (14a) exists for $x Ay$, then this expansion is given in (14).

Proof. By Lemma 6, $A \in T(A)$, and by Lemma 7, $U'(A) = U(A)$, $V'(A) = V(A)$.

PUNISHMENT, LIMBO, AND REWARD

When $U(A) \neq 0$ and $V(A) \neq 0$, A will be called a regular game. In this chapter, concepts of punishment, limbo, and reward are introduced for regular games. Homogeneity and convexity principles are found for these concepts which lead to simple criteria for choosing optimal strategies. Finally, use is made of these concepts to express the value and the optimal strategy sets for any regular game. It is assumed for the remainder of this chapter that A is regular.

Simple Games

Certain games become tractable immediately by means of the expression in Theorem 1.

Lemma 8. Let $XU(A) \neq 0$. Then $\Delta(A) \geq uAv$, $u \in U(A)$, $v \in V(A)$.

Let $YV(A) \neq 0$. Then $\Delta(A) \leq uAv$, $u \in U(A)$, $v \in V(A)$.

Proof. By the Main Theorem, there exists a $y^* \in Y^*$ such that for every $x \in X$,

$$\Delta(A) \geq xAy^*.$$

If $XU(A) \neq 0$, let $x = u \in X$, and use Theorem 1. Then,

$$\Delta(A) \geq uAy^* = uAv + (u - u)A(y^* - v) = uAv.$$

The outside members of this inequality give the first statement. A symmetric argument will show the statement for $YV(A) \neq 0$.

Theorem 2. Let $XU(A) \neq 0$, $YV(A) \neq 0$, and $u \in XU(A)$, $v \in YV(A)$. Then, $\Delta(A) = uAv$, $u \in X^\circ$, $v \in Y^\circ$.

Proof. Both statements of Lemma 8 are valid, and,

$$uAv \geq \Delta(A) \geq uAv, \text{ or } \Delta(A) = uAv.$$

By Theorem 1, for every $x \in X$, $y \in Y$,

$$uAy = uAv + (u - u)A(y - v) = uAv,$$

$$xAv = uAv + (x - u)A(v - v) = uAv.$$

Combine the last three equations and obtain,

$$uAy = \Delta(A) = xAv \text{ for every } x \in X, y \in Y.$$

This form demonstrates that $u \in X^\circ$, $v \in Y^\circ$.

The solution found in Theorem 2 (where the inequality signs on both sides are deleted) is called a simple solution. A game which has only simple solutions will be called a simple game. There exists a procedure which will find all solutions of an arbitrary game, by finding simple games derived from the arbitrary game, in a finite number of steps. The procedure is described in (2, pp. 27). When neither player has an optimal strategy with a zero component, the game is called a completely mixed game.

Theorem 3. Let A be a simple game. Then, $X^\circ = XU(A)$, $Y^\circ = YV(A)$, $\Delta(A) = uAv$, $u \in U(A)$, $v \in V(A)$.

Proof. By the definition of a simple game, and the Main Theorem, there exist X° , Y° such that if $x^\circ \in X^\circ$, $y^\circ \in Y^\circ$,

then for every $x \in X$, $y \in Y$,

$$x^\circ Ay = \Delta(A) = xAy^\circ.$$

In particular, this must hold when x and y are unit vectors.

Written in component form, the relation becomes,

$$x^\circ_1 a_{1j} = \Delta(A) = a_{1j} y^\circ_j, \text{ for all } i \text{ and } j.$$

But, as demonstrated in Lemma 7, these are equivalent definitions to (7') and (8') for U and V . Hence, $x^\circ \in U(A)$, $y^\circ \in V(A)$. Also, $x^\circ \in X$, $y^\circ \in Y$, and this means that $x^\circ \in XU(A)$, $y^\circ \in YV(A)$. By the Main Theorem, an x° and y° exist, hence, $XU(A) \neq 0$, $YV(A) \neq 0$. Then Theorem 2 applies, and the entire statement of the theorem follows immediately.

With the aid of the concepts of punishment, limbo, and reward, it will be easy to see that a completely mixed game is a simple game with unique elements in its U and V .

Artificial Games

Artificial games are devised in this section in order to study the structure of non-simple games more extensively. The definition of such an arbitrary game is suggested by a generalization of the Main Theorem.

Let M be an arbitrary m by m matrix, and consider the transformation,

$$z = xM, \quad x \in X.$$

Let the set of all such z be $X(M)$. Similarly, when N is an arbitrary n by n matrix, let the set of all z' defined by,

$$z' = Ny, y \in Y,$$

be $Y(N)$. $X(M)$ and $Y(N)$ are convex hyper-polyhedra of dimensions not more than their generators, X and Y .

Lemma 9. Let A be arbitrary, x have domain $X(M)$, y have domain $Y(N)$. Then,

$$\max_x \min_y xAy = \min_y \max_x xAy.$$

Proof. Let x' have domain X , y' have domain Y , $B = MAN$.

Then, apply Property 3C to B , x' , y' to obtain,

$$\begin{aligned} \max_x \min_y xAy &= \max_x \min_y x'MANy', \\ &= \max_x \min_y x'By', \\ &= \min_y \max_x x'By', \\ &= \min_y \max_x x'MANy', \\ &= \min_y \max_x xAy. \end{aligned}$$

For the present purpose, M and N will be taken such that X is a subset of $X(M)$, and Y is a subset of $Y(N)$.

Then, M and N must be non-singular, and satisfy,

$$Me = e, eN = e.$$

The entire proceeding can be regarded in two ways. First, the players keep their strategy sets, X and Y , but play a new game, $B = MAN$. Otherwise, the players alter their strategy sets to $X(M)$ and $Y(N)$, and play the original game,

A, over the new strategy sets. This chapter uses the latter point of view, along with conditions on M and N.

Let M be such that X is a subset of $X(M)$, but otherwise arbitrary. Let $N = I$, the identity matrix. Then $Y(N) = Y$. Call such an arbitrary game $A(M)$, and regard the game from the point of view of P_x . P_y is restricted to Y, but P_x has the additional freedom of $X(M)$. An examination of the game $A(M)$ can be made, and methods investigated by which P_x can reduce strategy choices from $X(M)$ to X with minimum loss.

In a symmetric way, a game, $A(N)$, is defined where $M = I$, and Y is a subset of $Y(N)$. Here, the additional freedom is given to P_y , and $A(N)$ becomes a tool for the investigation of A by P_y .

When P_x selects an $x \in X(M)$ in $A(M)$, the least favorable payoff will occur when P_y selects a $y^m \in Y$ such that,

$$x Ay^m = \text{Min}_y xAy, y^m = y^m(x).$$

P_x wishes to maximize the payoff, and will try to choose an $x^m \in X(M)$ such that,

$$\text{Min}_y x^m Ay = \text{Max}_x \text{Min}_y xAy.$$

If $x^m \in X$, an optimal strategy for A has been found. If $x^m \notin X$, then P_x must change his strategy when playing A. It turns out that certain criteria can be established in making the change from $X(M)$ to X.

The proceeding in $A(N)$ is similar, except that P_y wishes to minimize that payoff rather than maximize it.

Let $x \in X(M)$, $y \in Y$, $u \in U(A)$, $v \in V(A)$. Then,

$$\begin{aligned}\text{Min}_y xAy &= \text{Min}_y (uAv + (x - u)A(y - v)), \\ &= uAv + \text{Min}_y (x - u)A(y - v), \\ &= uAv + m(x),\end{aligned}$$

where,

$$m(x) = \text{Min}_y (x - u)A(y - v), \quad x \in X(M), \quad y \in Y. \quad (15)$$

Also, let $x \in X$, $y \in Y(N)$, $u \in U(A)$, $v \in V(A)$. Then,

$$\begin{aligned}\text{Max}_x xAy &= \text{Max}_x (uAv + (x - u)A(y - v)), \\ &= uAv + \text{Max}_x (x - u)A(y - v), \\ &= uAv - n(y),\end{aligned}$$

where (note minus sign),

$$n(y) = - \text{Max}_x (x - u)A(y - v), \quad x \in X, \quad y \in Y(N). \quad (16)$$

P_x will wish to select x to maximize $m(x)$, for this will maximize $\text{Min}_y xAy$. P_y will also want to maximize $n(y)$, for this will minimize $\text{Max}_x xAy$. The use of the minus sign in (16) makes the desires of the players with regard to their respective functions, $m(x)$ and $n(y)$, identical. By this means, then, a completely symmetrical description of the criteria for the two players will be possible. Note that the operators Max_x , Min_y are defined over X , Y respectively.

Definition 1. If $m(x) < 0$ or $n(y) < 0$, x or y is a point of punishment. If $m(x) = 0$ or $n(y) = 0$, x or y is a point of limbo. If $m(x) > 0$ or $n(y) > 0$, x or y is a point of reward.

Let P be the set of all points of punishment for either player, L be the set of all points of limbo, and R the set of all points of reward. Then for P_x , for example,

$$P + L + R = X(M), \quad PL = LR = RP = 0.$$

Let $P(\rho)$, a subset of P , be the set of all points with a punishment not more than ρ , and $R(\rho)$, a subset of R , the set of all points with a reward not less than ρ . Take $\rho > 0$ always in either case. Then for P_y , say,

$$\begin{aligned} y \in P(\rho) & \text{ implies } -\rho \leq n(y) < 0, \\ y \in L & \text{ implies } n(y) = 0, \\ y \in R(\rho) & \text{ implies } n(y) \geq \rho > 0. \end{aligned}$$

Homogeneity and Convexity in Punishment, Limbo, and Reward

If x is rewritten as $u + (x - u)$, it is easily seen that

$$u + \gamma(x - u), \quad \gamma > 0$$

is a half-ray with its origin at u in the direction $(x - u)$. The half-ray does not contain the origin, u . When $\gamma < 1$, a particular value, a point is represented by the expression which lies between u and x . If $\gamma > 1$, a particular value, the point lies beyond x from u .

Similarly, the expression,

$$v + \gamma(y - v), \quad \gamma > 0,$$

is a half-ray with the same connotation as described above.

Theorem 4. Let $x \in X(M)$, $u \in U$, $\gamma \geq 0$. Then,

$$m(u + \gamma(x - u)) = \gamma m(u + (x - u)) = \gamma m(x). \quad (17)$$

Let $y \in Y(N)$, $v \in V$, $\gamma \geq 0$. Then,

$$n(v + \gamma(y - v)) = \gamma n(v + (y - v)) = \gamma n(y). \quad (18)$$

Proof. By (15), $y \in Y$, and,

$$\begin{aligned} m(u + \gamma(x - u)) &= \text{Min}_y (u + \gamma(x - u) - u) A(y - v) \\ &= \text{Min}_y \gamma(x - u) A(y - v), \\ &= \gamma \text{Min}_y (x - u) A(y - v), \\ &= \gamma m(x) = \gamma m(u + (x - u)). \end{aligned}$$

The second statement follows a similar argument using (16).

Corollary 2. $UX(M)$ is a subset of L , $VY(N)$ is a subset of L .

Proof. Let $\gamma = 0$ in Theorem 4. Then $m(u) = 0$. A similar argument holds for $VY(N)$.

Theorem 4 establishes U and V as centers of homogeneity in punishment, limbo, and reward. By Corollary 2, U and V themselves contain only limbo points. If any point of a half-ray has the property of punishment, limbo, or reward, then every point of the half-ray has the same property. Both the punishment and the reward increase as the point moves away from the origin of the half-ray, u or v . From the point of view of either player, the punishment should be minimized, and the reward maximized. If a player is required to choose a strategy from a half-ray with origin u or v ,

Theorem 4 makes the choice an easy one. Since every point of a half-ray has the same property, the half-ray may be referred to as a punishment, limbo, or reward half-ray.

Criterion 1. Let a player's choice of strategy be restricted to a closed subset of a half-ray with origin u or v .

On a punishment half-ray, the player should select the strategy nearest u or v .

On a limbo half-ray, the player may select any strategy of the subset.

On a reward half-ray, the player should select the strategy farthest from u or v .

In particular, this criterion can be used in applying $A(M)$ or $A(N)$ to A by considering the intersection of a given half-ray with X or Y . If $u \notin X$ or $v \notin Y$, the intersection will be closed, and the criterion will be valid.

Points of a subset of E_m or E_n can be described in terms of a fixed set of vectors with origins u or v . For instance, consider the description of $x \in X$ in the following way.

Write the unit vectors of E_m as e^i . Then,

$$x = x_1 e^1 = u + (x_1 e^1 - u) = u + x_1 (e^1 - u).$$

From a practical standpoint, this fixed set has advantages in that all coefficients (the x_i) are positive, and that the property of each half-ray in the set is easy to calculate.

Lemma 10. Let $x \in X(M)$, $x' \in X(M)$, $0 \leq \alpha \leq 1$. Then,

$$m(\alpha x + (1 - \alpha)x') \geq \alpha m(x) + (1 - \alpha)m(x'). \quad (19)$$

Let $y \in Y(N)$, $y' \in Y(N)$, $0 \leq \alpha \leq 1$. Then,

$$n(\alpha y + (1 - \alpha)y') \geq \alpha n(y) + (1 - \alpha)n(y').$$

Proof. The proof of (19) will make use of (15), and the inequality,

$$\text{Min}_y(f(y) + g(y)) \geq \text{Min}_y f(y) + \text{Min}_y g(y).$$

Using these, one obtains,

$$\begin{aligned} m(\alpha x + (1 - \alpha)x') &= \text{Min}_y(\alpha x + (1 - \alpha)x' - u)A(y - v), \\ &= \text{Min}_y(\alpha(x - u)A(y - v) \\ &\quad + (1 - \alpha)(x' - u)A(y - v)), \\ &\geq \text{Min}_y \alpha(x - u)A(y - v) \\ &\quad + \text{Min}_y (1 - \alpha)(x' - u)A(y - v), \\ &\geq \alpha \text{Min}_y(x - u)A(y - v) \\ &\quad + (1 - \alpha) \text{Min}_y(x' - u)A(y - v), \\ &\geq \alpha m(x) + (1 - \alpha)m(x'). \end{aligned}$$

The statement of (20) can be shown similarly. It turns out that the operator $-\text{Max}_x$ gives the inequality in the same direction as Min_y . That is,

$$-\text{Max}_x(f(x) + g(x)) \geq (-\text{Max}_x f(x)) + (-\text{Max}_x g(x)).$$

By means of this lemma, the following two theorems characterize the sets, $R(\rho)$ and $P(\rho)$. The conditions for the characterization of $P(\rho)$ may seem especially stringent, but later results will support them.

Theorem 5. $R(\rho)$ is convex.

Proof. Let $x \in R(\rho)$, $x' \in R(\rho)$, $0 \leq \alpha \leq 1$. Then it must be shown that $\alpha x + (1 - \alpha)x' = x'' \in R(\rho)$. Use Lemma 12. Then,

$$\begin{aligned} m(\alpha x + (1 - \alpha)x') &\geq \alpha m(x) + (1 - \alpha)m(x'), \\ &\geq \alpha \rho + (1 - \alpha)\rho = \rho > 0. \end{aligned}$$

Hence, $x'' \in R(\rho)$ and $R(\rho)$ is convex.

Theorem 6. Let Z be a convex subset of X' or Y' , and in addition, let $ZP = Z$. Then, $ZP(\rho)$ is convex.

Proof. Let $x \in ZP(\rho)$, $x' \in ZP(\rho)$, $0 \leq \alpha \leq 1$. Then it must be shown that $\alpha x + (1 - \alpha)x' = x'' \in ZP(\rho)$. Use the convexity of Z , $ZP = Z$, and Lemma 12. Then,

$$\begin{aligned} 0 > m(\alpha x + (1 - \alpha)x') &\geq \alpha m(x) + (1 - \alpha)m(x') \\ &\geq -\alpha \rho - (1 - \alpha)\rho = -\rho. \end{aligned}$$

Hence, $x'' \in ZP(\rho)$, and $ZP(\rho)$ is convex.

Consider the set of all points with reward exactly ρ . By Theorem 5, this set is contained in a convex set. Consider one such point and by combining the statements of Theorem 4 and Corollary 1, notice that the point is not an element of U or V , hence there exists a half-ray with origin u or v through the point; then every neighborhood of this point contains a point of the half-ray not in $R(\rho)$, that is, every neighborhood of a point with reward exactly ρ contains points not in $R(\rho)$. Thus, the set of all points with reward

exactly ρ lies on the boundary of $R(\rho)$. On the other hand, if a point has reward different from ρ , say θ , this point lies on the boundary of $R(\theta)$, and is a finite distance along a half-ray from u or v from the boundary of $R(\rho)$. The boundary of $R(\rho)$ cannot lie along a half-ray for a finite distance, hence, the set of points with reward exactly ρ is the boundary of the convex set $R(\rho)$. Similarly, the set of all points of punishment exactly ρ is the boundary of $P(\rho)$, and under the conditions of Theorem 6, is the boundary of a convex set.

Let P_x consider all sets of equi-reward or equi-punishment points in $A(M)$. In $X(M)$, which is supposed to contain X , these sets may intersect X or not, depending on the magnitude of the punishment or reward. In view of Criterion 1, it is wished to select a strategy in X in such a way that every neighborhood of the strategy contains points not in X . On the other hand, if such a strategy is selected, and other points of its equi-reward or equi-punishment set have neighborhoods contained completely in X , then Theorem 4 says a greater reward or a less punishment can be obtained along a ray through u and the point. Hence, P_x should find a set, $R(\rho)$ or $P(\rho)$ which has a non-empty intersection with X , but which has no point with a neighborhood contained entirely in X . This set is, then, tangent to X . If the conditions of Theorem 6 hold, the

intersection of the sets in either case will be convex.

Criterion 2. Let either player be required to select a strategy from a convex subset of X or Y respectively, which contains either only points of punishment or only points of reward. Then, the player should select a strategy from the convex subset of X or Y which lies in the convex set of tangency between a set of equi-reward or equi-punishment and the given subset.

The completely mixed game can be disposed of at this time. Suppose A is completely mixed. Then no optimal strategy for either player has a zero component, i.e. no optimal strategy lies on the boundary of X or Y . This means the optimal strategies are not chosen from a punishment set or a reward set; hence they must be chosen from limbo sets of the two players. But these optimal strategies cannot be chosen from a limbo half-ray, for otherwise Criterion 1 says points on the boundaries of X and Y would also be optimal. Thus, the points must be chosen from U and V . Again, U and V must each have a unique element, for otherwise, Theorem 2 says points on the boundary would be optimal. Hence, the unique elements $u \in U(A)$, $v \in V(A)$ are optimal strategies, and the game is simple.

Solutions for Regular Games

The punishment, limbo, and reward sets belonging to the two players stem from two different considerations of the same matrix, A . Some of the interdependences involved in this are demonstrated here.

Lemma 11. Let the domain of x be X , the domain of y be Y . Then,

$$\text{Max}_X m(x) = - \text{Max}_Y n(y).$$

Proof. Use will be made of (3C), and the identity,

$$\text{Max}_X F(x) = - \text{Min}_X (-F(x)),$$

which can be verified on inspection. Recall in (15), (16), the variable killed by Max_X or Min_Y had domain X or Y . Then,

$$\begin{aligned} \text{Max}_X m(x) &= \text{Max}_X (\text{Min}_Y xAy - uAv), \\ &= \text{Max}_X \text{Min}_Y xAy - uAv, \\ &= \text{Min}_Y \text{Max}_X xAy - uAv, \\ &= \text{Min}_Y (uAv - n(y)) - uAv, \\ &= uAv + \text{Min}_Y (-n(y)) - uAv, \\ &= - \text{Max}_Y n(y). \end{aligned}$$

Lemma 12. $XR \neq 0$ if and only if $YP = Y$.

$YR \neq 0$ if and only if $XP = X$.

Proof. If $XR \neq 0$, then $\text{Max}_X m(x) > 0$, $x \in X$, and by Lemma 11, $\text{Max}_Y n(y) < 0$, $y \in Y$. Hence, Y contains only points of punishment, and $YP = Y$.

On the other hand, if $YP = Y$, Y contains only points of punishment, and $\text{Max}_Y n(y) < 0$, $y \in Y$. By Lemma 11 then, $\text{Max}_X m(x) > 0$, $x \in X$, and $XR \neq 0$. The second statement follows a symmetric argument.

Lemma 13. $XR = 0$ and $YR = 0$ if and only if $XL \neq 0$ and $YL \neq 0$.

Proof. Let $XR = 0$ and $YR = 0$. Then, $XL \neq 0$ or $XP = X$. The latter case is impossible, for if true, $YR \neq 0$, by Lemma 12, violating the hypothesis. Hence, $XL \neq 0$. By interchanging X and Y in the argument, $YL \neq 0$ also.

Let $XL \neq 0$ and $YL \neq 0$. Then $XR = 0$ or $XR \neq 0$. The latter case is impossible, for if true, $YP = Y$, and $YL = 0$, violating the hypothesis. Hence $XR = 0$. It follows, by interchanging X and Y , that $YR = 0$ also.

Theorem 7. Every regular game is in one and only one of the three classes defined by the conditions below.

- A. $XR = 0$ and $YR = 0$.
- B. $XR \neq 0$.
- C. $YR \neq 0$.

In addition, equivalent conditions for A, B, and C, respectively, are given in Lemma 12 and Lemma 13.

Proof. A, B, and C exhaust all possibilities except $XR \neq 0$ and $YR \neq 0$, which is impossible by Lemma 12. Hence, they exhaust all regular games. Their mutual exclusiveness can

be seen immediately by the forms of the conditions. Hence, every regular game must be in one and only one of the classes.

Classes B and C are essentially the same type with an interchange of players. If $XR \neq 0$, then, by Lemma 12, $YP = Y$.

Let Y be the convex set in the conditions of Theorem 6.

Then, $XR(\rho)$ and $YP(\rho)$ are both convex sets. By Criterion 2, each of the players should select a strategy from the convex set of tangency between X or Y and the equi-reward or equi-punishment set tangent to X or Y . Let the reward on the equi-reward set be ρ^* . Then, by Lemma 11, the punishment on the equi-punishment set is also ρ^* .

Theorem 8. Let A be regular, $u \in U(A)$, $v \in V(A)$, ρ^* the reward of an equi-reward set tangent to X or Y . Then, each of the exhaustive, but mutually exclusive, classes, given in Theorem 7, have values and solutions as given below.

A. If $XR = 0$ and $YR = 0$, then $\Delta(A) = uAv$,
 $X^* = XL$, $Y^* = YL$.

B. If $XR \neq 0$, then $\Delta(A) = uAv + \rho^*$,
 $X^* = XR(\rho^*)$, $Y^* = YP(\rho^*)$.

C. If $YR \neq 0$, then $\Delta(A) = uAv - \rho^*$,
 $X^* = XP(\rho^*)$, $Y^* = YR(\rho^*)$.

Proof. A. Let $XR = 0$ and $YR = 0$. Then $XL \neq 0$ and $YL \neq 0$.

Let $x^* \in XL$, $y^* \in YL$, then, for every $x \in X$, $y \in Y$,

$$x^{\circ}Ay \geq \min_y x^{\circ}Ay = uAv + m(x^{\circ}) = uAv,$$

$$xAy^{\circ} \leq \max_x xAy^{\circ} = uAv - n(y^{\circ}) = uAv.$$

Combine these inequalities, with the result,

$$x^{\circ}Ay \geq uAv \geq xAy^{\circ},$$

which displays statement A.

B. Let $x^{\circ} \in XR(\rho^{\circ})$, $y^{\circ} \in YP(\rho^{\circ})$. Then, for every $x \in X$, $y \in Y$,

$$x^{\circ}Ay \geq \min_y x^{\circ}Ay = uAv + m(x^{\circ}) \geq uAv + \rho^{\circ},$$

$$xAy^{\circ} \leq \max_x xAy^{\circ} = uAv - n(y^{\circ}) \leq uAv + \rho^{\circ}.$$

That is,

$$x^{\circ}Ay \geq uAv + \rho^{\circ} \geq xAy^{\circ},$$

which displays statement B.

C. Let $x^{\circ} \in XP(\rho^{\circ})$, $y^{\circ} \in YR(\rho^{\circ})$. Then, for every $x \in X$, $y \in Y$,

$$x^{\circ}Ay \geq \min_y x^{\circ}Ay = uAv + m(x^{\circ}) \geq uAv - \rho^{\circ},$$

$$xAy^{\circ} \leq \max_x xAy^{\circ} = uAv - n(y^{\circ}) \leq uAv - \rho^{\circ}.$$

That is,

$$x^{\circ}Ay \geq uAv - \rho^{\circ} \geq xAy^{\circ},$$

which displays statement C.

AN EQUIVALENCE RELATION FOR ALL GAMES

The operator, $T(W, \gamma, f, g)$, given in (13) is defined for an arbitrary game, A . It will be shown in this chapter that T divides all games of a given size into sets of games which are equivalent in a certain sense. Some of the invariants of these classes of equivalent games are investigated. In addition, the structural relationship between regular and irregular games is studied.

Equivalence of A and TA

Recall from (13) that,

$$B = TA = T(w_{ij}, \gamma, f_i, g_j)A = D + \gamma E + F + G$$

means that,

$$b_{ij} = w_{pq}(a_{ij} - a_{iq} - a_{pj} + a_{pq}) + \gamma + f_i + g_j,$$

where, $e_{pq}w_{pq} = 1$, and γ, f_i, g_j are completely arbitrary.

The operator $T(W', \gamma', f', g')$ will be written T' , and similarly T'' etc. If the operation is iterated, say $TT'A$, a typical entry of this element is

$$\begin{aligned} & w_{pq}(w'_{rs}(a_{ij} - a_{is} - a_{rj} + a_{rs}) + \gamma' + f'_i + g'_j \\ & - w'_{rs}(a_{iq} - a_{is} - a_{rq} + a_{rs}) - \gamma' - f'_i - g'_q \\ & - w'_{rs}(a_{pj} - a_{ps} - a_{rj} + a_{rs}) - \gamma' - f'_p - g'_j \\ & + w'_{rs}(a_{pq} - a_{ps} - a_{rq} + a_{rs}) + \gamma' + f'_p + g'_q) \\ & + \gamma + f_i + g_j, \end{aligned}$$

which reduces to,

$w_{pq}(a_{1j} - a_{1q} - a_{pj} + a_{pq}) + \gamma + f_1 + g_j$,
an entry of TA . That is,

$$TT'A = TA. \quad (21)$$

Definition 2. If for some T , $B = TA$, denote the relation between B and A by $B \text{ r } A$.

Theorem 9. $B \text{ r } A$ is an equivalence relation.

Proof. Reflexivity, symmetry, and transitivity must be shown for $B \text{ r } A$.

Reflexivity. Let T be selected by means of Lemma 6 in such a way that $TA = A$. Then, $A \text{ r } A$.

Symmetry. Let $B \text{ r } A$, then $B = TA$. Select T' such that $T'A = A$, and use (21). Then,

$$A = T'A = T'TA = T'B,$$

hence, $A \text{ r } B$.

Transitivity. Let $C \text{ r } B$, $B \text{ r } A$, then, $C = TB$, $B = T'A$. Choose T'' such that $T''A = A$. Then,

$$A = T''A = T''T'A = T''B = T''TB = T''C,$$

hence, $A \text{ r } C$.

Let a set of games of the same size be given, say A, A', A'', \dots . Then, by (21), a fixed T will classify these games by the alternatives $TA = TA'$ or $TA \neq TA'$. In the first case, A and A' are in the same equivalence set, $\overline{T}(A)$. In the

latter case, A and A' are not in the same equivalence set. On the other hand, by Lemma 6, an arbitrary game has a representation, $A = TA$ for some T . Hence, all games are divided into classes of games where all games of a single class are equivalent in the sense of Definition 2.

The relation between regular and irregular games is taken up now, and ultimately finds characterization in the arguments of T. Let $a \in X'$, $b \in Y'$ hereafter.

Lemma 14. Let A be arbitrary. Then there exists, and can be constructed, a $B = A + F + G$ which is regular.

Proof. By Lemma 5, $U(B) \neq 0$ if and only if there exists no t which satisfies,

$$(A + F + G)t = e, \quad et = 0.$$

Let k be fixed, and define g by the relation,

$$g_j = 1 - a_{kj} \text{ for all } j,$$

let l be fixed, and define f by the relation,

$$f_i = 1 - a_{il} \text{ for all } i.$$

Suppose a set of t exists for which $et = 0$. Then,

$$(a_{ij} + f_i + g_j)t_j = a_{ij}t_j + f_i e_j t_j + e_j t_j - a_{kj}t_j = 1.$$

If $et = 0$, this reduces to,

$$a_{ij}t_j - a_{kj}t_j = 1 \text{ for all } i.$$

This is clearly impossible when $i = k$, for the left side is then zero. In a similar way, the definition of f will guarantee that $V(B) \neq 0$.

Theorem 10. Let A be regular, $u \in U(A)$, $v \in V(A)$. Then,

$$A = T(u_p v_q, uAv, 0, 0)A.$$

Proof. Recall the construction of A in Lemma 6, and let

$w_{pq} = u_p v_q$. Then,

$$\begin{aligned} A &= T(u_i v_j, -u_p a_{pq} v_q, e_p u_p a_{iq} v_q, u_p a_{pj} e_q v_q)A, \\ &= T(u_i v_j, -uAv, a_{iq} v_q, u_p a_{pj})A. \end{aligned}$$

But by Lemma 7, the vectors in the last two arguments are uAv for all i or j , hence,

$$\begin{aligned} A &= T(u_i v_j, -uAv, uAve, uAve)A, \\ &= D - uAve + uAve + uAve, \\ &= D + uAve, \\ &= T(u_i v_j, uAv, 0, 0)A. \end{aligned}$$

Corollary 3. Let $T = T(a_1 b_j, \gamma, 0, 0)$. Then, $a \in U(TA)$,

$b \in V(TA)$, $\gamma = aTab$. (Consequently, TA is regular.)

Proof. Substitute a into (7'), with the result,

$$\begin{aligned} &a_1(a_p(a_{ij} - a_{iq} - a_{pj} + a_{pq})b_q + \gamma \\ &\quad - a_p(a_{ik} - a_{iq} - a_{pk} + a_{pq})b_q - \gamma) = 0, \end{aligned}$$

which reduces to,

$$a_1(a_{ij} - a_{ik}) - a_p(a_{pj} - a_{pk}) = 0.$$

Both terms on the left are identical, hence $a \in U(TA)$. The assertion for b can be verified in a similar way, and TA is regular. Then using (21), $TA = TTA$, one can apply Theorem 10.

Corollary 4. Let A be arbitrary. Then, A can be written,

$$A = T(u_1 v_j, u B v, f, g) A, \quad u \in U(B), \quad v \in V(B),$$

where B , f and g are defined in Lemma 14.

Proof. Use Lemma 14 and Theorem 10.

Invariants of $T(A)$

Consider the following relations,

$$u^\circ_i (a_{1j} - a_{1k}) = 0, \text{ for all } j, \text{ a fixed } k,$$

$$(a_{1j} - a_{kj}) v^\circ_j = 0, \text{ for all } i, \text{ a fixed } k,$$

$$e_i u^\circ_i = 0, \quad e_j v^\circ_j = 0,$$

and let the set of all u° satisfying these relations be U° ,
the set of all such v° be V° .

Theorem 11. $U^\circ(TA) = U^\circ(A)$, $V^\circ(TA) = V^\circ(A)$.

Proof. $U^\circ(TA)$ is defined by the following relations,

$$\begin{aligned} u^\circ_i (w_{pq}(a_{1j} - a_{1q} - a_{pj} + a_{pq}) + \gamma + f_1 + g_j \\ - w_{pq}(a_{1k} - a_{1q} - a_{pk} + a_{pq}) - \gamma - f_1 - g_k) = 0, \\ u^\circ_i e_{pq} w_{pq}(a_{1j} - a_{1k}) - e_i u^\circ_i e_q w_{pq}(a_{pj} - a_{pk}) \\ + e_i u^\circ_i (g_j - g_k) = 0. \end{aligned}$$

The last two terms are zero by definition of u° , and

$$u^\circ_i (a_{1j} - a_{1k}) = 0.$$

Hence, $u^\circ \in U^\circ(A)$. The reduction is reversible, hence, the first assertion is true. A symmetric argument will demonstrate the second statement.

Corollary 5. Let $u^\circ \in U^\circ(A)$, $v^\circ \in V^\circ(A)$, $T = T(a_i b_j, \gamma, 0, 0)$.

Then, $a + u^\circ = u \in U(TA)$, $b + v^\circ = v \in V(TA)$.

Proof. Both a and u° satisfy the homogeneous relations (7') and their sum will likewise satisfy the relations. Also, $e(a + u^\circ) = 1 + 0 = 1$, so u satisfies (10). A symmetric argument will prove the assertion for V .

It may occur to the reader that an extension of the equivalence classification, by dropping the requirement that $e_{pq} w_{pq} = 1$ can be effected. In particular, Theorem 9 and Theorem 11 do not need the restriction. In fact, the entire function of the restriction is to guarantee that the expansion will represent the original xAy , and not a multiple of xAy . In place of this extension, the reader is referred to the concepts in Property 3B.

In view of the facts noted previously, namely,

$$T(TA) = T(A), U^\circ(TA) = U^\circ(A), V^\circ(TA) = V^\circ(A),$$

none of these sets depend on A explicitly, except that A has generated a class of elements, TA . Let an index, π , be assigned to each class, and write,

$$T = T(\pi), U^\circ = U^\circ(\pi), V^\circ = V^\circ(\pi).$$

Thus, for a given π , there is a set of games, or matrices, and a subspace of each E_m and E_n in correspondence. By means of the results above, any regular game can be expanded as a function of π and T .

Theorem 12. Let $u^0 \in U^0(\pi)$, $v^0 \in V^0(\pi)$, $A \in T(\pi)$,

$T = T(a, b, \gamma, 0, 0)$. Then, for $x \in X'$, $y \in Y'$,

$$xTAy = \gamma + (x - a - u^0)A(y - b - v^0).$$

Proof. TA is regular by Corollary 3. The values of p , u , and v in Theorem 1 are given in Corollary 3 and Corollary 5.

Thus the expansion of every regular game is a function of π and T . γ has the effect of translating the value of the game, but alters the strategy considerations not at all. The essential character of all regular games in a class is then contained in the subset defined by $T = T(a_i, b_j, 0, 0, 0) = T_{ab}$. Call this subset $T^0(\pi)$.

Theorem 13. Every element of $T^0(\pi)$ has the same rank.

Proof. Let A be of minimum rank in $T(\pi)$, say of rank r .

Then, an entry of A can be written,

$$a_{ij} = \xi_{ik}h_{kj} \text{ where } k = 1, 2, \dots, r.$$

An entry of $B \in T^0(\pi)$ is,

$$\begin{aligned} b_{ij} &= a_p(\xi_{ik}h_{kj} - \xi_{ik}h_{kq} - \xi_{pk}h_{kj} + \xi_{pk}h_{kq})b_q, \\ &= a_p(\xi_{ik} - \xi_{pk})(h_{kj} - h_{kq})b_q, \\ &= g'_{ik}h'_{kj}. \end{aligned}$$

Now, B cannot be less than rank r , for otherwise A is not of minimum rank. On the other hand, B cannot be of rank more than r , by reason of the form displayed above. Hence B is of rank exactly r , and B is a typical element of $T^0(\pi)$.

Corollary 6. Let the minimum rank in $T(\pi)$ be r . Then, every element of $T(\pi)$ is of rank r , $r + 1$, or $r + 2$, not exceeding $\text{Min}(m, n)$, and every regular element of $T(\pi)$ is of rank r or $r + 1$. In addition,

$$r \leq \text{Min}(m, n) - 1.$$

Proof. Any element of $T(\pi)$ can be expressed by the method in Theorem 13 as, for a typical entry,

$$b_{ij} = g'_{ik} h'_{kj} + \gamma + f_i + g_j, \quad k = 1, 2, \dots, r.$$

Now γ can be absorbed in f or g , since both are arbitrary. Hence, any element is of rank at most $r + 2$. If B is regular, it has a representation with $f = 0$, $g = 0$, and hence, is of rank at most $r + 1$. For the last assertion, consider H_{pq} , defined in (6'), which is an element of $T^0(\pi)$, and has a zero row and a zero column.

Some Special Classes

Elements and properties of some special sets are displayed here to illustrate the analysis above.

$T(E)$, where the matrix E is taken as the index, is the set of all games with the typical entry,

$$a_{ij} = a_p b_q (1 - 1 - 1 + 1) + \gamma + f_i + g_j,$$

that is,

$$A = F + G,$$

where γE has been absorbed into the arbitrary F and G .

$T^0(E)$ contains only one element, the matrix 0, and,

$$U^0(E) = X', \quad V^0(E) = Y'.$$

The set of all regular games of the class is γE , and the set of all irregular games is $F + G$, where $f \neq \alpha e$ or $g \neq \beta e$. The minimum rank is zero, all regular elements are of rank zero or one, and the irregular elements are of rank one or two.

$T(I)$, where I is the identity matrix (square), is the set of all games with the typical entry, where $I = (\delta_{ij})$,

$$\begin{aligned} a_{ij} &= a_p b_q (\delta_{ij} - \delta_{iq} - \delta_{pj} + \delta_{pq}) + \gamma + f_i + g_j, \\ &= \delta_{ij} - b_i - a_j + ab + \gamma + f_i + g_j. \end{aligned}$$

That is,

$$A = I + F + G,$$

$T^0(I)$ contains all games of the form,

$$B = I - F^0 - G^0 + f^0 g^0 E,$$

where $f^0 \in X'$, $g^0 \in Y'$. The spaces associated with I are the zero vectors, written as,

$$U^0(I) = (0), \quad V^0(I) = (0).$$

$I \notin T^0(I)$, and is translated by $\gamma = 1/n$ from the game,

$$I - E/n \in T^0(I).$$

The minimum rank is $n - 1$, which demonstrates the attainment of the maximum given in the bound of Corollary 6.

METHODS AND CONJECTURES

Methods

Consider the game (2, p. 69),

$$A = \begin{vmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 12 \end{vmatrix}.$$

This game is regular, and the subspaces associated with its class are the zero vectors. That is, $U^0 = (0)$, $V^0 = (0)$. Then, there is a unique u or v . They turn out to be, on calculation by (7'), (8'), and (10),

$$u = v = (12/23, 12/23, -1/23), \quad uAv = -12/23.$$

It is easy to see that,

$$\max_i \min_j a_{ij} = 0,$$

and, since

$$uAv + m(x) = \min_y xAy,$$

$$uAv + \max_x m(x) = \max_x \min_y xAy \geq \max_i \min_j a_{ij} = 0.$$

That is,

$$\max_x m(x) \geq -uAv = 12/23,$$

hence, $RX \neq 0$, and by Lemma 12, $PY = Y$.

Consider the cone given by, $x = u + \gamma(z - u)$, where $z = (\alpha, 1 - \alpha, 0)$. By Theorem 4, $m(x) = \gamma m(z)$. X is a subset of this cone when $0 \leq \alpha \leq 1$, and $\gamma \geq 1$ for all points of X .

By definition,

$$\begin{aligned} m(z) &= \text{Min}_y zAy - uAv, \\ &= \text{Min}(3\alpha - 34/23, 35/23 - 3\alpha, 12/23). \end{aligned}$$

That is,

$$\begin{aligned} m(z) &= 3\alpha - 34/23 \text{ when } \alpha \leq 1/2, \\ m(z) &= 35/23 - 3\alpha \text{ when } \alpha \geq 1/2. \end{aligned}$$

The punishment, limbo, and reward sets can then be seen by comparing $m(z)$ with zero. This gives,

$$\begin{aligned} z \in P &\text{ when } \alpha < 34/69 \text{ and } \alpha > 35/69, \\ z \in L &\text{ when } \alpha = 34/69 \text{ and } \alpha = 35/69, \\ z \in R &\text{ when } 34/69 < \alpha < 35/69. \end{aligned}$$

The homogeneity of punishment, limbo, and reward makes it easy to see that these sets are cones with vertices at u . The limbo half-rays separate the punishment and reward sets, and serve as asymptotes for equi-reward or equi-punishment curves in the plane X' . By the symmetry, it is apparent that the point of tangency between X and an equi-reward set will be along the half-ray defined by $\alpha = 1/2$, out from u as far as possible, namely to $(0, 0, 1)$. Then,

$$\text{Max}_x m(x) = 12/23 = \rho^0.$$

Thus, $(0, 0, 1)$ is optimal, and $\Delta = uAv + \rho^0 = 0$.

Let $y = v + \gamma(z - v)$, z defined as above. Then, $n(y) = \gamma n(z)$. By calculation,

$$\begin{aligned} n(z) &= - \text{Max}_x xAz + uAv, \\ &= - \text{Max}(3\alpha - 34/23, 35/23 - 3\alpha, 12/23), \end{aligned}$$

which becomes,

$$n(z) = 3\alpha - 35/23 \text{ when } \alpha \leq 1/3,$$

$$n(z) = -12/23 \text{ when } 1/3 \leq \alpha \leq 2/3,$$

$$n(z) = 34/23 - 3\alpha \text{ when } \alpha \geq 2/3.$$

P_y has only punishing points, hence wants to stay as close to v as possible on any given half-ray from v . This set of points is z when $0 \leq \alpha \leq 1$, and by inspection above,

$$\text{Max}_z n(z) = -12/23 \text{ when } 1/3 \leq \alpha \leq 2/3,$$

and z so defined is optimal for P_y . It is interesting to note that the curves of equ-punishment in Y' are straight line segments inside the subset where $0 \leq \alpha \leq 1$, and curvilinear elsewhere.

Conjectures

The relation between regular and irregular games needs further examination, especially the role of F and G as functions of the solutions of the irregular games. One specific inquiry would be the following.

Question 1. Let A be irregular. When does there exist a B , F , and G such that B is regular, and,

$$A = B + F + G,$$

$$X^*(B)X^*(F) \neq 0, Y^*(B)Y^*(G) \neq 0?$$

If the intersection of either player listed were not zero, then the set would be optimal.

The analysis has treated an entire given game with no deletion of its components. However, certain games are either solvable on inspection, or can be simplified by easily seen deletions. It would be an advantage if these games could be recognized within the framework of the analysis.

A strictly determined game A is a game such that,

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij},$$

and is solvable on inspection. Every class, $T(\pi)$, has strictly determined games, for instance, H_{pq} , and these games may be regular or irregular. As examples, E , F , and G are strictly determined, but E is regular while F and G are irregular in general.

Question 2. Can the condition that a game be strictly determined be expressed entirely as a function of T ?

Symmetric games have taken on new importance recently (2, pp. 73), (2, pp. 81), with certain computational methods of solution, and new methods of symmetrizing arbitrary games. A symmetric game has a skew-symmetric payoff matrix and it is easily verified that $U = V$. Hence, if the payoff is A , $uAv = 0$, which means that if A is regular, then $A \in T^0(\pi)$. Then by Theorem 13, unless the minimum rank of a class is even

there exist no regular symmetric games in the class. It can be shown easily that a necessary and sufficient condition that a class have either a symmetric or skew-symmetric payoff matrix is that T_{aa} produce a symmetric or skew-symmetric matrix, respectively. This means, then, that a class cannot have both a symmetric and a skew-symmetric matrix.

Since the elements of a class play no special role in its definition, it seems that some particular T might be convenient to display an element of each class for comparison and identification purposes. With no basis to favor one a or b over another, it seems reasonable to let $a = e/m$, $b = e/n$. If this were done, this representative element would display the existence of symmetric or skew-symmetric elements in the class immediately.

LITERATURE CITED

1. von Neumann, John and Morgenstern, Oskar. Theory of Games and Economic Behaviour. Second edition, Princeton, New Jersey, Princeton University Press. 1947.
2. Tucker, A. W. and others. Contributions to the Theory of Games. Princeton, New Jersey, Princeton University Press. 1950.

ACKNOWLEDGMENT

I wish to express my sincere appreciation to Professor Bernard Vinograde for introducing me to the theory of games, and for his enthusiastic direction of this dissertation.